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13. ABSTRACT (Maximum 200 words)  This proposal addresses the development of the theory and implementation of algorithms for semidefinite programming (SDP) and also investigates the applications of SDP. The objectives of this research project consist of: 1) advancing the knowledge of the theory and implementation of second-order primal-dual methods for SDP; 2) developing primal-dual interior-point (IP) algorithms to solve more general classes of problems; 3) enhancing the variety, applicability, usefulness, and robustness of first-order nonlinear programming methods for SDP; 4) investigating the use of first-order methods for solving extremely large linear programs that are not efficiently solvable by the simplex method or IP methods; 5) developing SDP-based methods for solving combinatorial optimization problems; and 6) implementing each of these ideas to compare them with existing methods and also to provide software for the research community.  This research will lead to new and improved algorithms and codes to find exact or approximate solutions to optimization problems arising in diverse applications in industry, finance, science, and engineering .			
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Sincerely,

**Final Report for Grant No. N00014-03-1-0401**  
**February 1, 2003 – July 31, 2004**

**Semidefinite and Cone Programming:  
Theory, Implementation and Applications**

by

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**I. Summary of Completed Project**

This project addresses the development of the theory and implementation of primal-dual interior-point algorithms for semidefinite and cone programming problems. More specifically, its objectives consist in:

- 1) advancing the knowledge of the theory of polynomial and superlinear convergence analysis of primal-dual methods for SDP and cone programming;
- 2) studying the existence and asymptotic behavior of continuous trajectories for SDP;
- 3) developing superlinearly convergent higher-order primal-dual interior-point methods for SDP problems without strictly complementary solutions;
- 4) developing interior-point primal-dual algorithms to solve more general classes of cone programming problems;
- 5) developing new and/or improving existing algorithms and implementations for first-order NLP methods for SDP;
- 6) implementing these new methods and comparing them against existing SDP methods;
- 7) developing and implementing SDP-based heuristics for solving various NP-hard combinatorial optimization problems;
- 8) investigating and exploiting the special structure of SDP relaxations of combinatorial optimization problems; and
- 9) developing specialized SDP-based implicit enumeration methods for certain combinatorial optimization problems.

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During the duration of this project, a total of fourteen reports [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] have been written which acknowledge the grant N00014-03-1-0401. The works [1, 3, 4, 9, 12, 13, 14] have appeared or will appear in press: [4, 12, 13, 14] in *SIAM Journal on Optimization* and [1, 3, 9] in *Mathematical Programming*.

## II. Main Research

Papers [13, 14], written jointly with Takashi Tsuchiya, present new complexity results for interior-point algorithms for linear programming. They are related with topic 1 of the above list of goals.

Paper [14] presents a variant of Vavasis and Ye's layered-step path following primal-dual interior-point algorithm for linear programming. The algorithm is a predictor-corrector type algorithm which uses from time to time the least layered squares (LLS) direction in place of the affine scaling direction. It has the same iteration-complexity bound of Vavasis and Ye's algorithm, namely  $\mathcal{O}(n^{3.5} \log(\bar{\chi}_A + n))$  where  $n$  is the number of nonnegative variables and  $\bar{\chi}_A$  is a certain condition number associated with the constraint matrix  $A$ . Vavasis and Ye's algorithm requires explicit knowledge of  $\bar{\chi}_A$  (which is very hard to compute or even estimate) in order to compute the layers for the LLS direction. In contrast, our algorithm uses the affine scaling direction at the current iterate to determine the layers for the LLS direction, and hence does not require the knowledge of  $\bar{\chi}_A$ . A variant with similar properties and with the same complexity has been developed by Megiddo, Mizuno and Tsuchiya. However, their algorithm needs to compute  $n$  LLS directions on every iteration while ours computes at most one LLS direction on any given iteration.

Paper [13] presents a new iteration-complexity bound for the Mizuno-Todd-Ye predictor-corrector (MTY P-C) primal-dual interior-point algorithm for linear programming. The analysis of the paper is based on the important notion of crossover events introduced by Vavasis and Ye. For a standard form linear program  $\min\{c^T x : Ax = b, x \geq 0\}$  with decision variable  $x \in \mathbb{R}^n$ , we show that the MTY P-C algorithm started from a well-centered interior-feasible solution with duality gap  $n\mu_0$  finds an interior-feasible solution with duality gap less than  $n\eta$  in  $\mathcal{O}(n^2 \log(\log(\mu_0/\eta)) + n^{3.5} \log(\bar{\chi}_A^* + n))$  iterations, where  $\bar{\chi}_A^*$  is a scaling invariant condition number associated with the matrix  $A$ . More specifically,  $\bar{\chi}_A^*$  is the infimum of all the conditions numbers  $\bar{\chi}_{AD}$ , where  $D$  varies over the set of positive diagonal matrices. Under the setting of the Turing machine model, our analysis yields an  $\mathcal{O}(n^{3.5} L_A + n^2 \log L)$  iteration-complexity bound for the MTY P-C algorithm to find a primal-dual optimal solution, where  $L_A$  and  $L$  are the input sizes of the matrix  $A$  and the data

$(A, b, c)$ , respectively. In contrast, the classical iteration-complexity bound for the MTY P-C algorithm depends linearly on  $L$  instead of  $\log L$ .

Papers [1] presents a first-order NLP algorithms for solving large-scale semidefinite programming (SDP) problems. The distinguishing feature of the proposed algorithm is a change of variables that replaces the symmetric, positive semidefinite variable  $X$  of an SDP problem in standard form with a rectangular variable matrix  $R$  according to the factorization  $X = RR^T$ . The rank of the factorization, i.e., the number of columns of  $R$ , is chosen minimally so as to enhance computational speed while maintaining equivalence with the SDP problem. Fundamental results concerning the convergence of the algorithm are derived, and exceptional computational results on several large-scale test SDP problems are also presented. Our computational experiments have shown that our method substantially outperforms the other available algorithms (e.g., interior-point methods and the spectral bundle method) for solving large-scale SDP problems. For example, our method can now solve MAXCUT semidefinite relaxations of graphs containing 20,000 nodes in less than 10 minutes. On the other hand, the spectral bundle method takes more than 60 hours to solve these problems and the interior-point methods can not even perform the first iteration due to the large size of these problems. Our algorithm has been implemented in a code called SDP-LR, which can be download from the website <http://www.isye.gatech.edu/~monteiro/software>. It is related with topics 5 and 6 of the above list of goals.

Paper [9] surveys the most recent methods that have been developed for the solution of semidefinite programs. It first concentrates on the methods that have been primarily motivated by the interior point (IP) algorithms for linear programming, putting special emphasis in the class of primal-dual path-following algorithms. It then discusses methods that have been developed for solving large-scale SDP problems. These include first-order nonlinear programming (NLP) methods and more specialized path-following IP methods which use the (preconditioned) conjugate gradient or residual scheme to compute the Newton direction and the notion of matrix completion to exploit data sparsity. The P.I. gave a semi-plenary lecture at the 18th International Symposium on Mathematical Programming about the material of paper [9].

Solving systems of linear equations with “normal” matrices of the form  $AD^2A^T$  is a key ingredient in the computation of search directions for interior-point algorithms. Paper [12], establishes that a well-known basis preconditioner for such systems of

linear equations produces scaled matrices with uniformly bounded condition numbers as  $D$  varies over the set of all positive diagonal matrices. In particular, it is shown that when  $A$  is the node-arc incidence matrix of a connected directed graph with one of its rows deleted, then the condition number of the corresponding preconditioned normal matrix is bounded above by  $m(n - m + 1)$ , where  $m$  and  $n$  are the number of nodes and arcs of the network. This paper is related with topic 1 of the above list of goals.

Paper [10] considers a variant of a well-known long-step primal-dual infeasible IP algorithm for solving the linear program  $\min\{c^T x : Ax = b, x \geq 0\}$ ,  $A \in \mathbb{R}^{m \times n}$ , where the search directions are computed by using iterative linear solvers. More specifically, the preconditioner studied in [12] is used to precondition the normal coefficient matrix and the resulting preconditioned normal system of equations is solved by a standard iterative linear solver. It is shown that the number of (inner) iterations of the iterative linear solver at each (outer) iteration of the algorithm is bounded by a polynomial in  $m$ ,  $n$  and a certain condition number associated with  $A$ , while the number of outer iterations is bounded by  $\mathcal{O}(n^2 \log \epsilon^{-1})$ , where  $\epsilon$  is a given relative accuracy level. As a special case, it follows that the total number of inner iterations is polynomial in  $m$  and  $n$  for the minimum cost network flow problem. This paper is related with topic 1 of the above list of goals.

Paper [8] extends the above ideas to the context of the convex quadratic program  $\min\{c^T x + x^T Qx/2 : Ax = b, x \geq 0\}$ . The key in this extension is the introduction of an *augmented normal equation* (ANE) for determining the primal-dual search directions for the convex QP, which can be nicely preconditioned by using the approach studied in [12]. An approach to compute a suitable inexact primal-dual search direction is proposed and complexity results similar to the ones obtained for LP in [12] are also developed.

The conjugate gradient (CG) algorithm is well-known to have excellent theoretical properties for solving linear systems of equations  $Ax = b$  where the  $n \times n$  matrix  $A$  is symmetric positive definite. However, for extremely ill-conditioned matrices the CG algorithm performs poorly in practice. Paper [11] discusses an adaptive preconditioning procedure which improves the performance of the CG algorithm on extremely ill-conditioned systems. It introduces the preconditioning procedure by applying it first to the steepest descent algorithm. Then, the same techniques are extended to the CG algorithm, and convergence to an  $\epsilon$ -solution in  $\mathcal{O}(\log \det(A) + \sqrt{n} \log \epsilon^{-1})$  iterations is proven, where  $\det(A)$  is the determinant of the matrix.

In the series of papers [3, 4, 5, 7], the convergence of the weighted central paths

associated with two central path maps and the central path for a class of degenerate semidefinite programs are studied. As a result, we obtain a new error bound that might eventually be useful in the derivation of new superlinear convergence results for interior-point primal-dual semidefinite programming methods. These papers are related with topic 2 of the above list of goals.

### III. Related Research.

Papers [2, 6] are not too related with the main topics of this project but have been written during its duration, and hence acknowledge it.

Paper [2] considers the proximal point method with Bregman distance applied to linear programming problems, and study the dual sequence obtained from the optimal multipliers of the linear constraints of each subproblem. Convergence of this dual sequence is established, as well as convergence rate results for the primal sequence, for a suitable family of Bregman distances. These results are obtained by studying first the limiting behavior of a certain perturbed dual path and then the behavior of the dual and primal paths.

Paper [6] presents a modified nearly exact (MNE) method for solving low-rank trust region (LRTR) subproblem. The LRTR subproblem is to minimize a quadratic function, whose Hessian is a positive diagonal matrix plus an explicit low-rank update, subject to a Dikin-type ellipsoidal constraint, whose scaling matrix is positive definite and has the similar structure as the objective Hessian just described. The nearly exact (NE) method proposed by Moré and Sorensen [15] is properly modified to solve the LRTR subproblem by completely avoiding the computation of Cholesky factorizations of large-scale matrices. The resulting MNE method is quite efficient and robust to for computing NE solutions of large-scale LRTR subproblems. This method can be applied to solve some large-scale nonlinear programming problems.

### IV. Miscellaneous

During the duration of this project, a total of thirteen presentations on my research work were given at major conferences and universities. Among these presentations, the most influential one was a semi-plenary talk that the P.I. gave at the 18th International Symposium on Mathematical Programming in Copenhagen on August of 2003.

The P.I. has collaborated with the following investigators during the course of this grant: Sam Burer (Dept. of Management Sciences, University of Iowa), Joao X. da Cruz Neto (Universidade Federal do Piaui, Teresina, Brazil), Orizon P. Ferreira

(Universidade Federal de Goias, Goiania, Brazil), Arkadi Nemirovski (Technion University, Israel), and Takashi Tsuchiya (Institute of Statistical Mathematics, Japan).

The P.I. is currently serving as Associate Editor for three journals, namely: *Mathematical Methods of Operations Research* (2002-present), *INFORMS Journal on Computing* (2001-present) and *Mathematics of Operations Research* (2003-present). He is also serving as Chair of the INFORMS Optimization Society (2002-present).

This grant has partially supported one Ph.D. student, namely Zhaosong Lu, who is expected to graduate on the summer of 2005.

Finally, due to his widely cited research work, the P.I. joined in 2004 the list of ISI Highly Cited Researchers (see <http://www.isihighlycited.com>).

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